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LETTER TO THE EDITOR

The effect of multiplicative noise on the relaxation time of a real non-linear physical system: a comparison of experiment and theory for the random growing rate model (RGRM)

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Abstract. It is found that the theoretical behaviour of the relaxation time T for the random growing rate model (RGRM) under the influence of multiplicative noise of intensity Q (a monotonic decrease of T^{-1} towards a limiting value as $Q \to \infty$) differs markedly from that actually measured in electronic simulators of that system. A new electronic analogue experiment is described in which a distinct minimum in $T^{-1}(Q)$ has been observed for the first time, with clear evidence for an increase of $T^{-1}(Q)$ with Q at large Q in good qualitative agreement with an earlier analogue experiment. The discrepancy between experiment and the theoretical solutions of the (idealised) equation is attributed to the profound influence exerted by the very weak *additive* noise which must also, in some measure, always be present in a real physical system.

The often unexpected forms of behaviour exhibited by non-linear dynamical systems in noisy environments can frequently be accounted for in terms of suitable model equations in which the noise enters into one of the terms multiplicatively. Extensive discussions of a wide variety of examples taken from physics, chemistry, engineering, biology and other branches of science and technology may be found in the books by Horsthemke and Lefever (1984) and Risken (1985) and in the review by Faetti *et al* (1985). Inherent in attempts to understand such systems is, of course, the ever-present danger that the idealised stochastic differential equations analysed by the theory may differ in subtle, but crucially significant, respects from the reality that they are intended to model. In this letter, we address a curious conundrum that has arisen in connection with the effect of multiplicative noise on the relaxation time T of a particular stochastic non-linear system: the so-called random growing rate model (RGRM), also known as the Stratonovich model. We will show that a number of earlier, seemingly contradictory, analogue measurements and predictions of T can be reconciled if explicit account is also taken of the additive noise that must also, in some measure, always be present in a real physical system (Schenzle and Brand 1979).

The stochastic differential equation in question, first studied by Stratonovich (1967), may be written

$$\dot{x} = dx - bx^3 + \xi x \tag{1}$$

where ξ represents Gaussian white noise with autocorrelation function

$$\langle \xi(t)\xi(t')\rangle = = 2Q\delta(t - t') \tag{2}$$

and d and b are constants. With an appropriate choice of values for these constants, (1) represents an overdamped anharmonic oscillator moving in a double-well potential that can be used to model numerous bistable systems occurring in nature: in what follows all measurements and calculations will be normalised to the particular case of $d=b=1$. Our interest centres on the relaxation time of the system

$$T=\int_0^\infty c(s)\,\mathrm{d}s/c(o) \tag{3}$$

where

$$c(s)=\langle\delta x(t+s)\,\delta x(t)\rangle \tag{4}$$

and

$$\delta x(t)=x(t)-\langle x(t)\rangle. \tag{5}$$

The possible variation of T with Q has been subjected to close theoretical scrutiny by a number of workers in order, particularly, to determine whether or not the phenomenon of critical slowing down occurs near the noise intensity

$$Q=Q_c=d \tag{6}$$

where the maximum in the density at finite x vanishes and a singular maximum appears at $x=0$ (Schenzle and Brand 1979). The recent calculations by Jung and Risken (1985) have established beyond doubt that, in reality, T^{-1} falls monotonically with increasing Q from its deterministic ($Q\to 0$) value of 2, towards an asymptotic ($Q\to\infty$) limit of $2/\pi$. This result confirms the conclusions reached previously by Hernández-Machado *et al* (1984) and is in excellent qualitative accord with a digital simulation by Sancho *et al* (1982a); it is, however, in clear and unambiguous disagreement with an analogue simulation experiment at the University of Pisa described by Faetti *et al* (1984) which showed definite evidence for an *increase* of T^{-1} with Q in the region of large Q. In this letter, we report the outcome of a completely new analogue experiment, carried out at the University of Lancaster in an attempt to resolve these discrepant results. It was hoped, in particular, to establish whether or not the relaxation time of a real physical system can be expected to behave in the manner predicted by the theoretical solutions of (1).

The analogue experiment was based on the electronic circuit shown in outline form in figure 1. Its design philosophy and mode of operation, and the method employed for the computation of T from the measurements of $x(t)$, were essentially the same as described previously (Sancho *et al* 1985) in connection with an investigation of relaxation times for the general cubic bistable. It is important to note, however, that the technique differed in several important respects from that used for the earlier analogue experiment on the RGRM by Faetti *et al* (1984): the electronic circuit was entirely different from the double-integrator/minimum-component design used by them; the noise source was a high quality commercial instrument (Wandel and Goltermann, model RGI), rather than a modified linear feedback shift register (LFSR) device, and the autocorrelation functions were computed by means of a standard digital fast Fourier transform (FFT) method, rather than by the specially designed analogue correlator developed at Pisa. It should perhaps be emphasised that the role of the computer in the present work was purely as a data processor for analysis of $x(t)$; the operation of the analogue circuit was quite unaffected by whether or not the computer was connected.

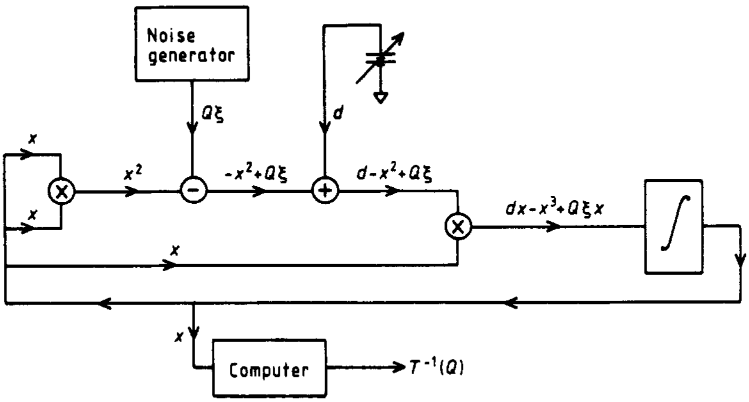

Figure 1. Block diagrams of the electronic circuit used to model (1). It is constructed entirely from standard analogue components. The computer serves as a data processor for the determination of reciprocal relaxation times $T^{-1}(Q)$ from the noise $x(t)$; it does not affect the operation of the circuit in any way.

Experimental measurements of $T^{-1}(Q)$, shown by the points of figure 2, were made for two different scalings of the circuit in order to accommodate a wide range of Q: the circles refer to $d=0.250$ and the squares to $d=0.125$. For more convenient comparison with each other, and with the theoretical predictions, the data have all been normalised by appropriate changes of variable to correspond to (1) with $d=b=1$. Estimates of their statistical reliability are indicated by the bars, except where these would have been smaller than the symbols themselves. The full curve represents the calculation of Jung and Risken (1985); it falls at large Q towards the asymptotic limit

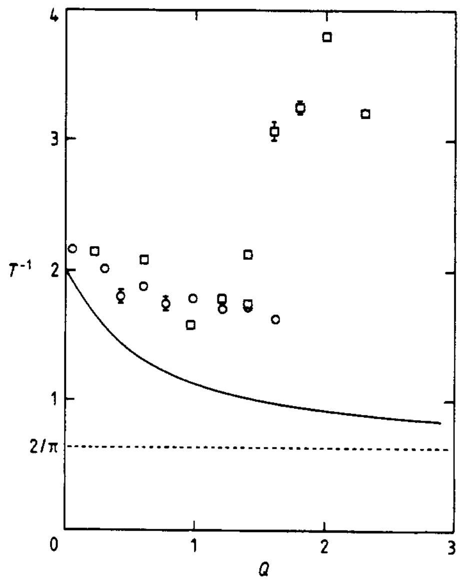

Figure 2. Reciprocal relaxation times T^{-1} measured for the circuit shown in figure 1 as a function of the noise intensity Q (points), compared with the calculation (full curve) of Jung and Risken (1985). The measurements, which were obtained for $d=0.25$ (circles) and $d=0.125$ (squares), have been normalised to the standard form (1) with $d=b=1$.

of $2/\pi$ (Hernández-Machado *et al* 1984) shown by the broken line. Three features of the data are of particular interest. First, and most important, although the experimental values of T^{-1} at first decrease with Q in qualitative agreement with the theoretical curve, they then pass through a distinct minimum and rise rapidly again at large Q. This behaviour is entirely consistent with the earlier measurements by Faetti *et al* (1984). Secondly, it is evident that, in their region of overlap, the circled data lie systematically below the squares. The third feature is rather more subtle but, as we shall see, is also of considerable significance. It is that the scatter of the data points about imaginary smooth curves drawn through them is much greater than can be accounted for purely in terms of their statistical uncertainties.

We have carried out complementary studies of $T^{-1}(Q)$ for (1) both by digital simulation and also through a calculation based on the projector operator method with an improved continued fraction procedure (CFP) that has enabled us to sum the entire Mori chain to infinity. In each case, the results obtained are in close agreement with the Jung and Risken (1985) curve of figure 2. (In addition, we have established that the earlier CFP calculation, reported by Faetti *et al* in 1984, yielded an incorrect result for $T^{-1}(Q)$ at large Q because of their truncation of the infinite Mori chain to 20 states.) These topics will be treated in detail in a forthcoming publication.

The question now needing to be resolved is this: why do calculations of $T^{-1}(Q)$ yield a monotonic decrease towards an asymptotic limit, whereas analogue experiments show distinct minima in $T^{-1}(Q)$, with an increase of T^{-1} with Q at large Q? We suggest that the answer is to be found in the weak *additive* noise that is always present in a real physical system (such as the analogue electronic circuit of figure 1). Following the ideas of Schenzle and Brand (1979) and Brand (1984), therefore, we propose that (1) has little connection with physical reality as it stands. For the equation to provide a satisfactory description of processes that occur in nature, it is essential that an additive noise term should also be included on the right-hand side. In the present situation, the extra term cannot be of zero mean, because transitions would then be able to take place between the positive and negative potential wells, a phenomenon that occurs with negligible frequency in practice. We conclude that (1) must be replaced by

$$\dot{x} = dx - bx^3 + \xi x + \zeta + g \tag{7}$$

where ζ represents noise of zero mean, which for convenience we take to be Gaussian and white, with autocorrelation function

$$\langle \zeta(t)\zeta(t')\rangle = 2D\delta(t-t') \tag{8}$$

and cross-correlation function

$$\langle \zeta(t)\xi(t')\rangle = 0 \tag{9}$$

and g is a constant that is small compared to the equilibrium value of x, $(d/b)^{1/2}$. In order to test this suggestion, we have recalculated $T^{-1}(Q)$ using the method of Jung and Risken (1985) but with (7) instead of (1), for a range of plausible values of D and g. Some results of these calculations are shown in figure 3. It is immediately evident that, even with $D = 0$, a small finite value of g gives rise to a minimum in $T^{-1}(Q)$. In practice, prior to each measurement of T^{-1}, g was set carefully to zero within experimental error, but, because the latter is finite, and because of thermal drifts occurring during the acquisition period, g will not have been *precisely* zero. For the $d = 0.125$ scaling of the circuit, we estimated that g usually lies in the range $0 < |g| < 5\times10^{-2}$, so that the corresponding measurements would be expected to fall between

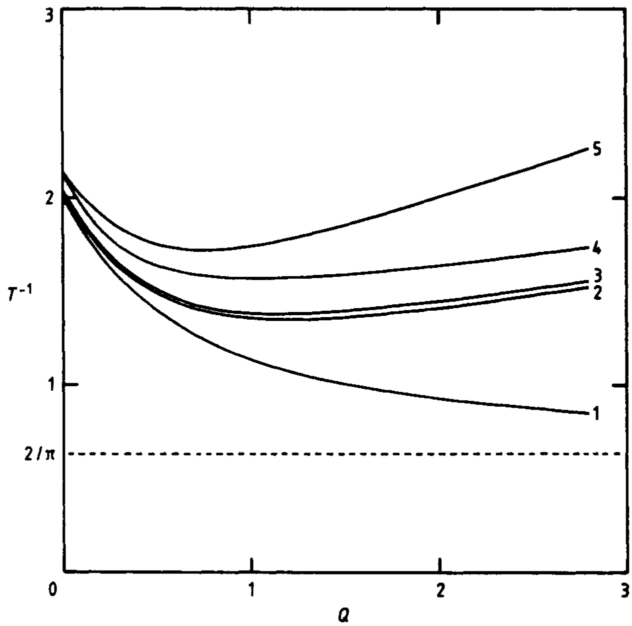

Figure 3. Reciprocal relaxation times T^{-1} as functions of the (multiplicative) noise intensity Q, calculated by the method of Jung and Risken (1985) for various values of D and g: curve (1) $D=0$, $g=0$; (2) $D=0$, $g=10^{-2}$; (3) $D=10^{-5}$, $g=10^{-2}$; (4) $D=0$, $g=5\times10^{-2}$; (5) $D=10^{-5}$, $g=5\times10^{-2}$.

curves 2 and 4. In reality (cf figure 2) the data lie much higher than this. When account is also taken of additive noise, however, the calculated values of T^{-1} are dramatically increased, even for extremely small values of D, as shown by curves 3 and 5 of figure 3 which are clearly in much better accord with the analogue measurements. Precise quantitative agreement is not, of course, to be anticipated because the effective value of D changes with x and consequently may be expected to increase with Q. For $Q=0$, the measured value of D was about 10^{-5} for $d=0.125$. It would appear, therefore, that the existence of the minimum in the $T^{-1}(Q)$ measurements of figure 2 and the earlier observation by Faetti *et al* (1984) of an increasing $T^{-1}(Q)$ in the high Q region can be accounted for in terms of weak additive noise within the analogue circuits (Faetti *et al* 1983) in conjunction with a small additive constant. The unexpectedly large scatter of the data may be attributed to the differing effective values of g for the different data points, which will have the effect illustrated by the shifts between curves 2 and 4, or 3 and 5, of figure 3.

In conclusion, we would make three observations. Firstly, it might appear surprising in the light of the above discussion that good agreement should have been obtained for $T^{-1}(Q)$ in the case of the general cubic bistable (Sancho *et al* 1985), without any consideration having been given to the possible influence of weak additive noise. There were important differences, however, between that system and the one currently under scrutiny: the minimum in $T^{-1}(Q)$ observed by Sancho *et al* was associated with interstate switching processes that do not occur in the present case; and the effect of the additive noise (that must, of course, also have been present in the cubic circuit) would have been relatively unimportant because there were no (unphysical) singularities to be removed (Schenzle and Brand 1979, Sancho *et al* 1982b) from the theoretical densities. Secondly, and closely related to the above comments, it would seem that the effect of (even very weak) additive noise on the relaxation time corresponds to the removal (Brand 1984) of the long-time tail (Brenig and Banai 1982) in the correlation function. Finally, in the light of the foregoing results and remarks, we may conclude that, although the monotonic decrease of $T^{-1}(Q)$ predicted theoretically (Sancho *et al* 1982a, Hernández-Machado *et al* 1984, Jung and Risken 1985) on the basis of (1) is formally correct, it represents a phenomenon that does not occur in practice in real physical systems.

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